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On Nash-Coalitive Fuzzy Continuous Differential Games

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***Abstract*— This paper concentrates on providing a simple and effective technique to solve the Nash-coalitive fuzzy continuous differential games. These games are featured by having fuzzy coefficients, fuzzy controls, and fuzzy state functions. The proposed technique utilizes the α-cut set concept of fuzzy numbers to transform the problem under study into a corresponding interval problem. The latter one is sliced into two problems, lower problem and upper problem, to obtain the optimal solution for the Nash-coalitive fuzzy continuous differential games problem in a range form. The sufficient and necessary conditions for this problem are derived. A numerical example is provided to clarify the presented concepts.**

***Keywords*— fuzzy control, fuzzy differential games, fuzzy numbers, Nash-coalitive differential games.**

I. INTRODUCTION

R. Isaacs [1] was the first to introduce and discuss the differential games. One of the main types of these games is Nash-coalitive continuous differential game (NCDG), a nonzero-sum {N+M}-person continuous differential game, in which N players establish a coalition and play against the remaining M players. In real-life, players may face the issue of the lack of information concerning the parameters of the differential game problem. One of the paramount ways to overcome this issue is to employ the fuzzy set or rough set theories.

Numerous researchers have studied the rough games, for instance, the authors of [2-4] have dealt with rough matrix games. In [5], an approach has been developed to solve the non-cooperative continuous differential games that include rough interval objectives and constraints. Moreover, Ammar et al. [6] have introduced a technique to treat the two-person zero-sum continuous differential games in rough interval environment.

Other researchers have considered the fuzzy games. For instance, triangular fuzzy payoffs matrix game has been treated using ranking technique in [7]. Trapezoidal payoffs matrix game has been discussed and solved in [8].

The authors of [9] have used a new similarity measure to analyze dual hesitant fuzzy matrix games. In [10], a fuzzy matrix games approach has been implemented for intelligent water management. Youness et al. [11] have derived the optimality conditions for non-coalitive fuzzy differential games. The authors of [12] have discussed the two-person zero-sum differential games in which the state trajectories and controls are fuzzy. Megahed et al. [13] have studied optimality of fuzzy control differential games with nonlinear membership function. In [14], the authors have utilized the Nash-coalitive approach to study the cooperation of governments to fight terrorists in a counterterrorism problem. Youness et al. [15] have analyzed the NCDG with parameters in the cost functions.

This paper presents the solution to the problem of the Nash-coalitive fuzzy continuous differential games (NCFDGs) that involves a coalition of N players playing against M players outside the coalition. The players inside the coalition cooperate to find their optimal controls to minimize their cost functional, and the M players do the same. The NCFDGs have fuzzy coefficients, fuzzy state trajectories and fuzzy controls. The main idea of the proposed technique is to decompose the problem into two problems, lower and upper problems, by using the concept of α-cut set. The necessary and sufficient conditions of the problem of NCFDGs are derived. These optimality conditions can be used to get the optimal solutions of the two aforementioned problems.

This paper consists of 6 sections; in section 2, the NCFDG problem is formulated, and its optimality conditions are derived in section 3. The proposed algorithm is clearly outlined in section 4. Section 5provides a numerical example which is treated by using our approach. Finally, section 6 renders conclusion and summary of this paper.

II. PROBLEM FORMULATION AND BASIC CONCEPTS

The Nash-coalitive fuzzy continuous differential game (NCFDG) problem, in which state trajectories and controls are all fuzzy, can be expressed as follows:

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NCFDG: Find $(\tilde{\eta}^*(t), \tilde{\vartheta}^*(t))$ which solves the problems:

$$(I): \min_{\tilde{\vartheta}}\{\sum_{j=1}^{M} \omega'_j \tilde{J}'_j\ (\tilde{\eta}^*(t), \tilde{\vartheta}(t)) = \sum_{j=1}^{M} \omega'_j \tilde{\psi}'_j\ [\tilde{r}(t_F)] +$$

$$\int_{t_o}^{t_F} \sum_{j=1}^{M} \omega'_j \tilde{K}'_j\ [t, \tilde{r}(t), \tilde{\eta}^*(t), \tilde{\vartheta}(t)]\ dt\}$$

subject to

$$\dot{\tilde{r}}(t) = h(t, \tilde{r}(t), \tilde{\eta}^*(t), \tilde{\vartheta}(t)), \qquad \tilde{r}(t_o) = \tilde{r}_o\ ,\ t \in [t_o, t_F]$$

and

$$(II): \min_{\tilde{\eta}}\{\sum_{i=1}^{N} \omega_i \tilde{J}_i(\tilde{\eta}(t), \tilde{\vartheta}^*(t)) = \sum_{i=1}^{N} \omega_i \tilde{\psi}_i [\tilde{r}(t_F)] +$$

$$\int_{t_o}^{t_F} \sum_{i=1}^{N} \omega_i \tilde{K}_i [t, \tilde{r}(t), \tilde{\eta}(t), \tilde{\vartheta}^*(t)]\ dt\}$$

subject to

$$\dot{\tilde{r}}(t) = h(t, \tilde{r}(t), \tilde{\eta}(t), \tilde{\vartheta}^*(t)), \qquad \tilde{r}(t_o) = \tilde{r}_o\ ,\ t \in [t_o, t_F]$$

where $\omega' \in W', W' = \{\omega' \in R^M \mid \omega'_j \geq 0, \sum_{j=1}^{M} \omega'_j = 1\}$ and

$\omega \in W, W = \{\omega \in R^N \mid \omega_i \geq 0, \sum_{i=1}^{N} \omega_i = 1\}$,

$\tilde{r}(t): [t_o, t_F] \to R^n$ is the fuzzy state vector of the game (the game starting is at $t = t_o$ and it ends at $t = t_F$),$[t_o, t_F]$ is the fixed predefined duration of the game, $\tilde{r}(t_o) = \tilde{r}_o$ is the initial state that all players know. $\tilde{\vartheta}(t) \in V \subset R^m$ denotes the fuzzy control vector of players outside the coalition and $\tilde{\eta}(t) \in Y \subset R^s$ denotes the fuzzy control vector of players inside the coalition, they are piecewise continuous fuzzy functions of time,

$\tilde{h}(.): [t_o, t_F] \times R^n \times R^m \times R^s \to R^n, s = \sum_{i=1}^{N} s_i, m = \sum_{j=1}^{M} m_j,$

$\tilde{K}_i(.): [t_o, t_F] \times R^n \times R^m \times R^s \to R^n$ is C^1, $i = 1, ..., N$,

$\tilde{K}'_j(.): [t_o, t_F] \times R^n \times R^m \times R^s \to R^n$ is C^1, $j = 1, ..., M$,

$\tilde{\psi}_i(.): R^n \to R,$ is C^1, $i = 1, ..., N$,

$\tilde{\psi}'_j(.): R^n \to R,$ is C^1, $j = 1, ..., M$,

$\tilde{J}_i, \tilde{J}'_j$ are the fuzzy cost functions of player (i) and player (j), respectively.

Definition 1 [16, 17]:

The α-cut set of a triangular fuzzy number $\tilde{B} = (b_1, b_2, b_3)$ is $(\tilde{B})_\alpha =$ $[(\tilde{B})^L_\alpha, (\tilde{B})^U_\alpha]$, where $(\tilde{B})^L_\alpha = (1-\alpha) b_1 + \alpha b_2$ and $(\tilde{B})^U_\alpha = (1-\alpha) b_3 + \alpha b_2$ represent the lower and upper bounds, respectively, $\alpha \in [0,1]$.

Definition 2 [16, 17]:

Let $\tilde{B}$ and $\tilde{C}$ be two fuzzy numbers, and $(\tilde{B})_\alpha =$ $[(\tilde{B})^L_\alpha, (\tilde{B})^U_\alpha]$ and $(\tilde{C})_\alpha =$ $[(\tilde{C})^L_\alpha, (\tilde{C})^U_\alpha]$ be their α-cut sets, respectively. We can define the multiplication of $(\tilde{B})_\alpha$ and $(\tilde{C})_\alpha$ as:

$$(\tilde{B})_\alpha * (\tilde{C})_\alpha = [min((\tilde{B})^L_\alpha * (\tilde{C})^L_\alpha, (\tilde{B})^L_\alpha * (\tilde{C})^U_\alpha, (\tilde{B})^U_\alpha * (\tilde{C})^L_\alpha,$$
$$(\tilde{B})^U_\alpha * (\tilde{C})^U_\alpha), max((\tilde{B})^L_\alpha * (\tilde{C})^L_\alpha, (\tilde{B})^L_\alpha * (\tilde{C})^U_\alpha, (\tilde{B})^U_\alpha * (\tilde{C})^L_\alpha,$$
$$(\tilde{B})^U_\alpha * (\tilde{C})^U_\alpha))].$$

By applying the concept of α-cut, $\alpha \in [0,1]$, the problem NCFDG can be transformed into the following interval problem:

α-NCFDG:

$$(I): \min_{(\tilde{\vartheta})_\alpha}\{\sum_{j=1}^{M} \omega'_j [(\tilde{J}'_j\ (\tilde{\eta}^*, \tilde{\vartheta})^L_\alpha, (\tilde{J}'_j\ (\tilde{\eta}^*, \tilde{\vartheta}))^U_\alpha] =$$

$$\sum_{j=1}^{M} \omega'_j [(\tilde{\psi}'_j\ (\tilde{r}(t_F))^L_\alpha, (\tilde{\psi}'_j\ (\tilde{r}(t_F))^U_\alpha] +$$

$$\int_{t_o}^{t_F} (\sum_{j=1}^{M} \omega'_j [(\tilde{K}'_j\ (t, \tilde{r}(t), \tilde{\eta}^*(t), \tilde{\vartheta}(t)))^L_\alpha, (\tilde{K}'_j\ (t, \tilde{r}(t), \tilde{\eta}^*(t), \tilde{\vartheta}(t)))^U_\alpha]) dt\}$$

subject to

$$[(\dot{\tilde r}(t))^L_\alpha,(\dot{\tilde r}(t))^U_\alpha]=[(\tilde h(t,\tilde r(t),\tilde\eta^*(t),\tilde\vartheta(t)))^L_\alpha,(\tilde h(t,\tilde r(t),\tilde\eta^*(t),\tilde\vartheta(t)))^U_\alpha],$$

$$[(\tilde r(t_o))^L_\alpha,(\tilde r(t_o))^U_\alpha]=[(\tilde r_o)^L_\alpha,(\tilde r_o)^U_\alpha], t\in[t_o,t_F].$$

$$(II):\min_{(\tilde\eta)_\alpha}\{\sum_{i=1}^{N}\omega_i[(\tilde J_i(\tilde\eta,\tilde\vartheta^*))^L_\alpha,(\tilde J_i(\tilde\eta,\tilde\vartheta^*))^U_\alpha]=$$

$$\sum_{i=1}^{N}\omega_i[(\tilde\psi_i(\tilde r(t_F)))^L_\alpha,(\tilde\psi_i(\tilde r(t_F)))^U_\alpha]+$$

$$\int_{t_o}^{t_F}(\sum_{i=1}^{N}\omega_i[(\tilde K_i(t,\tilde r(t),\tilde\eta(t),\tilde\vartheta^*(t)))^L_\alpha,(\tilde K_i(t,\tilde r(t),\tilde\eta(t),\tilde\vartheta^*(t)))^U_\alpha])dt\}$$

subject to

$$[(\dot{\tilde r}(t))^L_\alpha,(\dot{\tilde r}(t))^U_\alpha]=[(\tilde h(t,\tilde r(t),\tilde\eta(t),\tilde\vartheta^*(t)))^L_\alpha,(\tilde h(t,\tilde r(t),\tilde\eta(t),\tilde\vartheta^*(t)))^U_\alpha],$$

$$[(\tilde r(t_o))^L_\alpha,(\tilde r(t_o))^U_\alpha]=[(\tilde r_o)^L_\alpha,(\tilde r_o)^U_\alpha], t\in[t_o,t_F].$$

The problem α -NCFDG can be split into the following two problems, namely, lower problem and upper problem:

Lower Problem:

$$(I):\min_{(\tilde\vartheta)^L_\alpha}\{\sum_{j=1}^{M}\omega'_j(\tilde J'_j(\tilde\eta^*,\tilde\vartheta))^L_\alpha=\sum_{j=1}^{M}\omega'_j(\tilde\psi'_j(\tilde r(t_F)))^L_\alpha+$$

$$\int_{t_o}^{t_F}(\sum_{j=1}^{M}\omega'_j(\tilde K'_j(t,\tilde r(t),\tilde\eta^*(t),\tilde\vartheta(t)))^L_\alpha dt\}$$

subject to

$$(\dot{\tilde r}(t))^L_\alpha=(\tilde h(t,\tilde r(t),\tilde\eta^*(t),\tilde\vartheta(t)))^L_\alpha,\ (\tilde r(t_o))^L_\alpha=(\tilde r_o)^L_\alpha, t\in[t_o,t_F].$$

$$(II):\min_{(\tilde\eta)^L_\alpha}\{\sum_{i=1}^{N}\omega_i(\tilde J_i(\tilde\eta,\tilde\vartheta^*))^L_\alpha=\sum_{i=1}^{N}\omega_i(\tilde\psi_i(\tilde r(t_F)))^L_\alpha+$$

$$\int_{t_o}^{t_F}(\sum_{i=1}^{N}\omega_i(\tilde K_i(t,\tilde r(t),\tilde\eta(t),\tilde\vartheta^*(t)))^L_\alpha dt\}$$

subject to

$$(\dot{\tilde r}(t))^L_\alpha=(\tilde h(t,\tilde r(t),\tilde\eta(t),\tilde\vartheta^*(t)))^L_\alpha,\ (\tilde r(t_o))^L_\alpha=(\tilde r_o)^L_\alpha, t\in[t_o,t_F].$$

Upper Problem:

$$(I):\min_{(\tilde\vartheta)^U_\alpha}\{\sum_{j=1}^{M}\omega'_j(\tilde J'_j(\tilde\eta^*,\tilde\vartheta))^U_\alpha=\sum_{j=1}^{M}\omega'_j(\tilde\psi'_j(\tilde r(t_F)))^U_\alpha+$$

$$\int_{t_o}^{t_F}(\sum_{j=1}^{M}\omega'_j(\tilde K'_j(t,\tilde r(t),\tilde\eta^*(t),\tilde\vartheta(t)))^U_\alpha dt\}$$

subject to

$$(\dot{\tilde r}(t))^U_\alpha=(\tilde h(t,\tilde r(t),\tilde\eta^*(t),\tilde\vartheta(t)))^U_\alpha,\ (\tilde r(t_o))^U_\alpha=(\tilde r_o)^U_\alpha, t\in[t_o,t_F].$$

$$(II):\min_{(\tilde\eta)^U_\alpha}\{\sum_{i=1}^{N}\omega_i(\tilde J_i(\tilde\eta,\tilde\vartheta^*))^U_\alpha=\sum_{i=1}^{N}\omega_i(\tilde\psi_i(\tilde r(t_F)))^U_\alpha+$$

$$\int_{t_o}^{t_F}(\sum_{i=1}^{N}\omega_i(\tilde K_i(t,\tilde r(t),\tilde\eta(t),\tilde\vartheta^*(t)))^U_\alpha dt\}$$

subject to

$$(\dot{\tilde r}(t))^U_\alpha=(\tilde h(t,\tilde r(t),\tilde\eta(t),\tilde\vartheta^*(t)))^U_\alpha,\ (\tilde r(t_o))^U_\alpha=(\tilde r_o)^U_\alpha, t\in[t_o,t_F].$$

Definition 3:

For the NCFDG problem, the point $((\tilde\eta^*)_\alpha,(\tilde\vartheta^*)_\alpha)$ is called an α -Nash-coalitive optimal (α -NCO) solution at (r_o,t_o) if and only if, for all $(\tilde\eta(t))_\alpha=(\tilde\eta_1(t),\tilde\eta_2(t),...,\tilde\eta_N(t))_\alpha$ and $(\tilde\vartheta)_\alpha(t)=(\tilde\vartheta_1(t),\tilde\vartheta_2(t),...,\tilde\vartheta_M(t))_\alpha$, we have:

$$(\tilde J_i((\tilde\eta^*_1,\tilde\eta^*_2,...,\tilde\eta^*_N),(\tilde\vartheta^*_1,\tilde\vartheta^*_2,...,\tilde\vartheta^*_M)))_\alpha\le$$
$$(\tilde J_i((\tilde\eta_1,\tilde\eta_2,...,\tilde\eta_N),(\tilde\vartheta^*_1,\tilde\vartheta^*_2,...,\tilde\vartheta^*_M)))_\alpha, i=1,2,...,N,$$

and

$$(\tilde J'_j((\tilde\eta^*_1,\tilde\eta^*_2,...,\tilde\eta^*_N),(\tilde\vartheta^*_1,\tilde\vartheta^*_2,...,\tilde\vartheta^*_M)))_\alpha\le$$
$$(\tilde J'_j((\tilde\eta^*_1,\tilde\eta^*_2,...,\tilde\eta^*_N),(\tilde\vartheta_1,\tilde\vartheta_2,...,\tilde\vartheta_M)))_\alpha, j=1,2,...,M.$$

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III. Necessary And Sufficient Conditions

Theorem 1 (Necessary Conditions):

Let $(\tilde{K}_i\ (t,\tilde{r}(t),\tilde{\eta}(t),\tilde{\vartheta}^*(t)))_\alpha^U$, $(\tilde{h}(t,\tilde{r}(t),\tilde{\eta}(t),\tilde{\vartheta}^*(t)))_\alpha^U$ and $(\tilde{K}'_j\ (t,\tilde{r}(t),\tilde{\eta}^*(t),\tilde{\vartheta}(t)))_\alpha^U$ be continuous differentiable functions. Assume that the upper problem has an open-loop α -NCO solution $((\tilde{\eta}^*)_\alpha^U,(\tilde{\vartheta}^*)_\alpha^U)$ with a corresponding state trajectory $(\tilde{r}^*(t))_\alpha^U$, then there exist scalars $\omega'_j \geq 0$, $j=1,...,M$, $\omega_i \geq 0$, $i=1,...,N$, and continuous costate functions $(\tilde{p})_\alpha^U(t):[t_o,t_F]\to R^n$ and $(\tilde{q})_\alpha^U(t):[t_o,t_F]\to R^n$ corresponding to Hamiltonians of problems (I) and (II) as in the form:

$$(\tilde{H}'(t,\tilde{r},\tilde{\eta}^*,\tilde{\vartheta},\omega'_j,\tilde{p}))_\alpha^U = \sum_{j=1}^{M}\omega'_j(\tilde{K}'_j\ (t,\tilde{r},\tilde{\eta}^*,\tilde{\vartheta}))_\alpha^U + ((\tilde{p})_\alpha^U)^T.(\tilde{h}(t,\tilde{r},\tilde{\eta}^*,\tilde{\vartheta}))_\alpha^U \tag{1}$$

and

$$(\tilde{H}(t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*,\omega_i,\tilde{q}))_\alpha^U = \sum_{i=1}^{N}\omega_i(\tilde{K}_i\ (t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*))_\alpha^U + ((\tilde{q})_\alpha^U)^T.(\tilde{h}(t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*))_\alpha^U \tag{2}$$

satisfying the following relations,

$$(\dot{\tilde{p}}(t))_\alpha^U = -\frac{\partial(\tilde{H}'(t,\tilde{r},\tilde{\eta}^*,\tilde{\vartheta},\omega'_j,\tilde{p}))_\alpha^U}{\partial(\tilde{r}(t))_\alpha^U} \tag{3}$$

$$(\tilde{p}(t_F))_\alpha^U = \sum_{j=1}^{M}\omega'_j\frac{\partial(\tilde{\psi}'_j\ (\tilde{r}(t_F)))_\alpha^U}{\partial(\tilde{r}(t_F))_\alpha^U} \tag{4}$$

$$(\dot{\tilde{q}}(t))_\alpha^U = -\frac{\partial(\tilde{H}(t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*,\omega_i,\tilde{q}))_\alpha^U}{\partial(\tilde{r}(t))_\alpha^U} \tag{5}$$

$$(\tilde{q}(t_F))_\alpha^U = \sum_{i=1}^{N}\omega_i\frac{\partial(\tilde{\psi}_i\ (\tilde{r}(t_F)))_\alpha^U}{\partial(\tilde{r}(t_F))_\alpha^U} \tag{6}$$

$$(\dot{\tilde{r}}(t))_\alpha^U = (\tilde{h}(t,\tilde{r}(t),\tilde{\eta}(t),\tilde{\vartheta}^*(t)))_\alpha^U, (\tilde{r}(t_o))_\alpha^U = (\tilde{r}_o)_\alpha^U, t\in[t_o,t_F] \tag{7}$$

$$\frac{\partial(\tilde{H}'(t,\tilde{r},\tilde{\eta}^*,\tilde{\vartheta},\omega'_j,\tilde{p}))_\alpha^U}{\partial(\tilde{\vartheta}(t))_\alpha^U} = 0 \tag{8}$$

$$\frac{\partial(\tilde{H}(t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*,\omega_i,\tilde{q}))_\alpha^U}{\partial(\tilde{\eta}(t))_\alpha^U} = 0 \tag{9}$$

Proof: For problem (II) in the upper problem, since the functions $(\tilde{K}_i\ )_\alpha^U$, $(\tilde{K}'_j\ )_\alpha^U$ and $(\tilde{h})_\alpha^U$ are continuous differentiable, then there exists a function $(\tilde{q}(t))_\alpha^U$ that is a solution to the differential equation:

$$(\dot{\tilde{q}}(t))_\alpha^U = -(\tilde{q}(t))_\alpha^U\frac{\partial(\tilde{h})_\alpha^U}{\partial(\tilde{r})_\alpha^U} - \frac{\partial}{\partial(\tilde{r})_\alpha^U}(\sum_{i=1}^{N}\omega_i(\tilde{K}_i\ )_\alpha^U) \tag{10}$$

such that

$$(\tilde{q}(t_F))_\alpha^U = \sum_{i=1}^{N}\omega_i\frac{\partial(\tilde{\psi}_i\ (\tilde{r}(t_F)))_\alpha^U}{\partial(\tilde{r}(t_F))_\alpha^U} \tag{11}$$

The adjoint equation of (10) is:

$$\left.\begin{aligned}&(\tilde{q}(t))_\alpha^U\delta(\dot{\tilde{r}}(t))_\alpha^U = \\ &[(\tilde{q}(t))_\alpha^U\frac{\partial(\tilde{h})_\alpha^U}{\partial(\tilde{r})_\alpha^U} + \frac{\partial}{\partial(\tilde{r})_\alpha^U}(\sum_{i=1}^{N}\omega_i(\tilde{K}_i\ )_\alpha^U)]\delta(\tilde{r}(t))_\alpha^U + \\ &(\tilde{H}(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*,\omega_i,\tilde{q}))_\alpha^U - (\tilde{H}(t,\tilde{r}^*,\tilde{\eta},\tilde{\vartheta}^*,\omega_i,\tilde{q}))_\alpha^U\end{aligned}\right\} \tag{12}$$

The solution of (12) is $\delta(\tilde{r}(t))_\alpha^U$ that has the initial condition $\delta(\tilde{r}(t_o))_\alpha^U = 0$, then from theorem 10.1 in [18], we get:

$$\begin{aligned}&\frac{d((\tilde{q}(t))_\alpha^U\delta(\tilde{r}(t))_\alpha^U)}{dt} = \\ &(\dot{\tilde{q}}(t))_\alpha^U\delta(\tilde{r}(t))_\alpha^U + (\tilde{q}(t))_\alpha^U\delta(\dot{\tilde{r}}(t))_\alpha^U = \\ &(\tilde{H}(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*,\omega_i,\tilde{q}))_\alpha^U - (\tilde{H}(t,\tilde{r}^*,\tilde{\eta},\tilde{\vartheta}^*,\omega_i,\tilde{q}))_\alpha^U\end{aligned} \tag{13}$$

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By integrating both sides, we obtain:

$$\left.\begin{array}{l}(\tilde{q}(t_F))^U_\alpha \delta(\tilde{r}(t_F))^U_\alpha = \int_{t_o}^{t_F}[(\tilde{H}(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*,\omega_i,\tilde{q}))^U_\alpha - \\ (\tilde{H}(t,\tilde{r}^*,\tilde{\eta},\tilde{\vartheta}^*,\omega_i,\tilde{q}))^U_\alpha]dt\end{array}\right\} \quad (14)$$

From (6), we have:

$$\left.\begin{array}{l}\dfrac{\partial\sum_{i=1}^{N}\omega_i(\tilde{J}_i)^U_\alpha}{\partial(\tilde{r}(t_F))^U_\alpha}\delta(\tilde{r}(t_F))^U_\alpha = \\ \sum_{i=1}^{N}\omega_i\dfrac{\partial(\tilde{\psi}_i(\tilde{r}(t_F)))^U_\alpha}{\partial(\tilde{r}(t_F))^U_\alpha}\delta(\tilde{r}(t_F))^U_\alpha = \delta(\tilde{J}_i(\tilde{\eta},\tilde{\vartheta}^*))^U_\alpha\end{array}\right\} \quad (15)$$

Then using theorem 11.1 in [18], equation (14) takes the form:

$$\delta(\tilde{J}_i(\tilde{\eta},\tilde{\vartheta}^*))^U_\alpha =$$

$$\int_{t_o}^{t_F}[(\tilde{H}(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*,\omega_i,\tilde{q}))^U_\alpha - (\tilde{H}(t,\tilde{r}^*,\tilde{\eta},\tilde{\vartheta}^*,\omega_i,\tilde{q}))^U_\alpha]dt \quad (16)$$

Let the sets $(V)^U_\alpha$ and $(Y)^U_\alpha$ be convex, then from differentiability of the functions $(\tilde{K}_i)^U_\alpha$, $(\tilde{K}'_j)^U_\alpha$ and $(\tilde{h})^U_\alpha$, and using theorem 11.2 in [18], we have:

$$[(\tilde{H}(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*,\omega_i,\tilde{q}))^U_\alpha - (\tilde{H}(t,\tilde{r}^*,\tilde{\eta},\tilde{\vartheta}^*,\omega_i,\tilde{q}))^U_\alpha] \le 0 \quad (17)$$

This means that

$$(\tilde{H}(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*,\omega_i,\tilde{q}))^U_\alpha = \min_{(\tilde{\eta})^U_\alpha}(\tilde{H}(t,\tilde{r}^*,\tilde{\eta},\tilde{\vartheta}^*,\omega_i,\tilde{q}))^U_\alpha \quad (18)$$

Similarly, for problem (I), we can easily prove that: $(\tilde{H}'(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*,\omega'_j,\tilde{p}))^U_\alpha = \min_{(\tilde{\vartheta})^U_\alpha}(\tilde{H}'(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta},\omega'_j,\tilde{p}))^U_\alpha$

By the same way, we can derive the necessary optimality conditions for the lower problem.

Theorem 2 (Sufficient Condition): The pair $((\tilde{\eta}^*)^U_\alpha,(\tilde{\vartheta}^*)^U_\alpha)$ is a NCO solution to problem (II) in the upper problem if there exist a function $(\tilde{q}(t))^U_\alpha$ for some scalars $\omega'_j \ge 0$ and $\omega_i \ge 0$ such that:

(i) $(\tilde{H}(t,\tilde{r}^*,\tilde{\eta},\tilde{\vartheta}^*,\omega_i,\tilde{q}))^U_\alpha - (\tilde{H}(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*,\omega_i,\tilde{q}))^U_\alpha +$
$(\dot{\tilde{q}}(t))^U_\alpha[(\tilde{r}(t))^U_\alpha - (\tilde{r}^*(t))^U_\alpha] \ge 0$ (19)

(ii) $(\tilde{q}(t_o))^U_\alpha[(\tilde{r}(t_o))^U_\alpha - (\tilde{r}^*(t_o))^U_\alpha] -$
$(\tilde{q}(t_F))^U_\alpha[(\tilde{r}(t_F))^U_\alpha - (\tilde{r}^*(t_F))^U_\alpha] \ge 0$ (20)

for all $(\tilde{r})^U_\alpha,(\tilde{\eta})^U_\alpha$ and $t \in [t_o,t_F]$.

Proof: Let $(\tilde{r}^*)^U_\alpha$ and $(\tilde{\eta}^*)^U_\alpha$ be admissible functions. Then, Hamiltonian function $(\tilde{H})^U_\alpha$ has the form:

$$(\tilde{H}(t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*,\omega_i,\tilde{q}))^U_\alpha = \sum_{i=1}^{N}\omega_i(\tilde{K}_i(t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*))^U_\alpha + ((\tilde{q})^U_\alpha)^T.(\tilde{h}(t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*))^U_\alpha \quad (21)$$

Using conditions (i) and (7), we obtain:

$$\left.\begin{array}{l}\sum_{i=1}^{N}\omega_i(\tilde{K}_i(t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*))^U_\alpha + ((\tilde{q})^U_\alpha)^T.(\tilde{h}(t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*))^U_\alpha - \\ \sum_{i=1}^{N}\omega_i(\tilde{K}_i(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*))^U_\alpha - ((\tilde{q})^U_\alpha)^T.(\tilde{h}(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*))^U_\alpha \\ +(\dot{\tilde{q}}(t))^U_\alpha[(\tilde{r}(t))^U_\alpha - (\tilde{r}^*(t))^U_\alpha] \ge 0\end{array}\right\} \quad (22)$$

which can be written as:

$$\sum_{i=1}^{N}\omega_i(\tilde{K}_i(t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*))^U_\alpha - \sum_{i=1}^{N}\omega_i(\tilde{K}_i(t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*))^U_\alpha + \frac{d}{dt}[(\tilde{q}(t))^U_\alpha[(\tilde{r}(t))^U_\alpha - (\tilde{r}^*(t))^U_\alpha]] \ge 0 \quad (23)$$

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By integrating (23), we have:

$$\left.\begin{aligned}&\int_{t_o}^{t_F}[\sum_{i=1}^{N}\omega_i(\tilde{K}_i\ (t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*))_\alpha^U-\sum_{i=1}^{N}\omega_i(\tilde{K}_i\ (t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*))_\alpha^U]dt\\&+(\tilde{q}(t_F))_\alpha^U[(\tilde{r}(t_F))_\alpha^U-(\tilde{r}^*(t_F))_\alpha^U]-\\&\quad(\tilde{q}(t_o))_\alpha^U[(\tilde{r}(t_o))_\alpha^U-(\tilde{r}^*(t_o))_\alpha^U]\geq 0.\end{aligned}\right\}\quad(24)$$

Then using condition (ii), we get:

$$\int_{t_o}^{t_F}[\sum_{i=1}^{N}\omega_i(\tilde{K}_i\ (t,\tilde{r},\tilde{\eta},\tilde{\vartheta}^*))_\alpha^U-\sum_{i=1}^{N}\omega_i(\tilde{K}_i\ (t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*))_\alpha^U]dt\geq 0$$

Similarly, for problem (I), we can prove that:

$$\int_{t_o}^{t_F}[\sum_{j=1}^{M}\omega_j'(\tilde{K}_j'\ (t,\tilde{r},\tilde{\eta}^*,\tilde{\vartheta}))_\alpha^U-\sum_{j=1}^{M}\omega_j'(\tilde{K}_j'\ (t,\tilde{r}^*,\tilde{\eta}^*,\tilde{\vartheta}^*))_\alpha^U]dt\geq 0\,.$$

Hence, $((\tilde{\eta}^*)_\alpha^U,(\tilde{\vartheta}^*)_\alpha^U)$ is an NCO solution to upper problem.

By the same way, the sufficient condition for lower problem can be proved.

IV. The Proposed Algorithm

The following steps can recapitulate the proposed technique for solving the NCFDG problem:

Step 1: Express the uncertainty in the game by using fuzzy coefficients and controls to formulate the NCFDG problem.

Step 2: Using the concept of α-cut, the NCFDG problem can be transformed into an interval optimization problem α-NCFDG.

Step 3: Divide (break down) problem α-NCFDG into two corresponding problems, upper problem and lower problem.

Step 4: Formulate the Hamiltonian functions of upper problem (1) - (2), and those of lower problem.

Step 5: Construct the equations (3) - (6) for upper problem and lower problem to get the costate variables.

Step 6: Find the candidate solutions for upper problem and lower problem by examining the necessary conditions (7)-(9).

Step 7: Finally, by satisfying the sufficient conditions, the α-NCO controls $(\tilde{\eta}^*)_\alpha=[(\tilde{\eta}^*)_\alpha^L,(\tilde{\eta}^*)_\alpha^U],(\tilde{\vartheta}^*)_\alpha=[(\tilde{\vartheta}^*)_\alpha^L,(\tilde{\vartheta}^*)_\alpha^U]$ and the corresponding state trajectory $(\tilde{r}^*)_\alpha=[(\tilde{r}^*)_\alpha^L,(\tilde{r}^*)_\alpha^U]$ are obtained.

V. Illustrative Example

Let the state equation be:

$\dot{\tilde{r}}(t)=\tilde{1}\tilde{\eta}_1(t)+\tilde{1}\tilde{\eta}_2(t)+\tilde{1}\tilde{\eta}_3(t)$, so there are three players in the game having the following respective cost functionals:

$$\tilde{J}_1=\tilde{1}\tilde{r}(t_F)-\int_{t_o}^{t_F}(\tilde{1}\tilde{\eta}_1(t)-\tilde{2})^2dt,$$

$$\tilde{J}_2=\tilde{2}\tilde{r}(t_F)-\int_{t_o}^{t_F}(\tilde{2}\tilde{\eta}_2(t)-\tilde{3})^2dt$$

$$and\ \tilde{J}_3=\tilde{2}\tilde{r}(t_F)-\int_{t_o}^{t_F}(\tilde{1}\tilde{\eta}_3(t)-\tilde{3})^2dt$$

Assume that the fuzzy coefficients are the following triangular fuzzy numbers: $\tilde{1}$=(0,1,2), $\tilde{2}$= (1,2,3) and $\tilde{3}$=(2,3,4). By definition 1, they have the following α -cut sets: $(\tilde{1})_\alpha=[\alpha,2-\alpha]$, $(\tilde{2})_\alpha=[1+\alpha,3-\alpha]$ and

$(\tilde{3})_\alpha=[2+\alpha,4-\alpha]$. Let $\tilde{r}(t_o)=0$.

For this game, let the second and third players form a coalition trying to minimize their costs and play against the first player who tries to minimize his cost as well. Accordingly, $\tilde{\vartheta}=\tilde{\eta}_1$ and $\tilde{\eta}=(\tilde{\eta}_2,\tilde{\eta}_3)$.

Now, the problem takes the form:

$$(I):\min_{\tilde{\vartheta}=\tilde{\eta}_1}\{\tilde{J}_1\}$$

$$(II):\min_{\tilde{\eta}_2,\tilde{\eta}_3}\{\tilde{J}=\omega_1\tilde{J}_2+\omega_2\tilde{J}_3\}$$

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subject to the state equation given above, and let $\omega_1 = \omega_2 = 0.5$.

Solution:

Using the concept of α-cut (let $\alpha = 0.5$), the given problem can be transformed into the following 0.5-NCFDG problem:

$$(I): \min_{(\tilde{\eta}_1)_{0.5}} \{[(\tilde{J}'_j)^L_{0.5}, (\tilde{J}'_j)^U_{0.5}] = [0.5,1.5][(\tilde{r}(t_F))^L_{0.5}, (\tilde{r}(t_F))^U_{0.5}] - \int_{t_o}^{t_F} ([0.5,1.5][(\tilde{\eta}_1)^L_{0.5}, (\tilde{\eta}_1)^U_{0.5}] - [1.5,2.5])^2 dt\},$$

$$(II): \min_{(\tilde{\eta}_2)_{0.5}, (\tilde{\eta}_3)_{0.5}} \{[(\tilde{J})^L_{0.5}, (\tilde{J})^U_{0.5}] = 0.5([1.5,2.5][(\tilde{r}(t_F))^L_{0.5}, (\tilde{r}(t_F))^U_{0.5}] + [2.5,3.5][(\tilde{r}(t_F))^L_{0.5}, (\tilde{r}(t_F))^U_{0.5}] - \int_{t_o}^{t_F} (([1.5,2.5][(\tilde{\eta}_2)^L_{0.5}, (\tilde{\eta}_2)^U_{0.5}] - [2.5,3.5])^2 + ([0.5,1.5][(\tilde{\eta}_3)^L_{0.5}, (\tilde{\eta}_3)^U_{0.5}] - [2.5,3.5])^2) dt)\}.$$

subject to

$$[(\dot{\tilde{r}}(t))^L_{0.5}, (\dot{\tilde{r}}(t))^U_{0.5}] = [0.5,1.5][(\tilde{\eta}_1)^L_{0.5}, (\tilde{\eta}_1)^U_{0.5}] + [0.5,1.5][(\tilde{\eta}_2)^L_{0.5}, (\tilde{\eta}_2)^U_{0.5}] + [0.5,1.5][(\tilde{\eta}_3)^L_{0.5}, (\tilde{\eta}_3)^U_{0.5}]$$

By definition 2, the above problem is sliced into two problems as follows:

Lower problem:

$$(I): \min_{(\tilde{\eta}_1)^L_{0.5}} \{[(\tilde{J}'_j)^L_{0.5}] = [0.5(\tilde{r}(t_F))^L_{0.5} - \int_{t_o}^{t_F} (1.5(\tilde{\eta}_1)^L_{0.5} - 1.5)^2 dt\}$$

$$(II): \min_{(\tilde{\eta}_2)^L_{0.5}, (\tilde{\eta}_3)^L_{0.5}} \{(\tilde{J})^L_{0.5} = 0.5(1.5(\tilde{r}(t_F))^L_{0.5} + 2.5(\tilde{r}(t_F))^L_{0.5} - \int_{t_o}^{t_F} ((2.5(\tilde{\eta}_2)^L_{0.5} - 2.5)^2 + (1.5(\tilde{\eta}_3)^L_{0.5} - 2.5)^2) dt)\}$$

subject to

$$(\dot{\tilde{r}}(t))^L_{0.5} = 0.5(\tilde{\eta}_1)^L_{0.5} + 0.5(\tilde{\eta}_2)^L_{0.5} + 0.5(\tilde{\eta}_3)^L_{0.5}$$

So, the Hamiltonian functions are:

$$(\tilde{H}')^L_{0.5} = -(1.5(\tilde{\eta}_1)^L_{0.5} - 1.5)^2 + 0.5(\tilde{p})^L_{0.5}((\tilde{\eta}_1)^L_{0.5} + (\tilde{\eta}_2)^L_{0.5} + (\tilde{\eta}_3)^L_{0.5}),$$

$$(\tilde{H})^L_{0.5} = -0.5[(2.5(\tilde{\eta}_2)^L_{0.5} - 2.5)^2 + (1.5(\tilde{\eta}_3)^L_{0.5} - 2.5)^2] + 0.5(\tilde{q})^L_{0.5}((\tilde{\eta}_1)^L_{0.5} + (\tilde{\eta}_2)^L_{0.5} + (\tilde{\eta}_3)^L_{0.5})$$

For problem (I): Since $(\dot{\tilde{p}}(t))^L_{0.5} = -\dfrac{\partial(\tilde{H}')^L_{0.5}}{\partial(\tilde{r})^L_{0.5}} = 0$, then $(\tilde{p}(t))^L_{0.5} = k$ (constant), and since $(\tilde{p}(t_F))^L_{0.5} = \dfrac{\partial(0.5(\tilde{r}(t_F))^L_{0.5})}{\partial(\tilde{r}(t_F))^L_{0.5}} = 0.5$, then $(\tilde{p}(t))^L_{0.5} = 0.5$.

Since $\dfrac{\partial(\tilde{H}')^L_{0.5}}{\partial(\tilde{\eta}_1)^L_{0.5}} = -2 \times 1.5(1.5(\tilde{\eta}_1)^L_{0.5} - 1.5) + 0.5(\tilde{p}(t))^L_{0.5} = 0$, then $(\tilde{\eta}_1)^L_{0.5} = 1.0556$.

For problem (II): Since $(\dot{\tilde{q}}(t))^L_{0.5} = -\dfrac{\partial(\tilde{H})^L_{0.5}}{\partial(\tilde{r})^L_{0.5}} = 0$, then $(\tilde{q}(t))^L_{0.5} = k$ (constant), and since $(\tilde{q}(t_F))^L_{0.5} = \dfrac{\partial(2(\tilde{r}(t_F))^L_{0.5})}{\partial(\tilde{r}(t_F))^L_{0.5}} = 2$, then $(\tilde{q}(t))^L_{0.5} = 2$.

Since $\dfrac{\partial(\tilde{H})^L_{0.5}}{\partial(\tilde{\eta}_2)^L_{0.5}} = -2.5(2.5(\tilde{\eta}_2)^L_{0.5} - 2.5) + 0.5(\tilde{q}(t))^L_{0.5} = 0$, then $(\tilde{\eta}_2)^L_{0.5} = 1.16$.

Similarly $\dfrac{\partial(\tilde{H})^L_{0.5}}{\partial(\tilde{\eta}_3)^L_{0.5}} = -1.5(1.5(\tilde{\eta}_3)^L_{0.5} - 2.5) + 0.5(\tilde{q}(t))^L_{0.5} = 0$, then $(\tilde{\eta}_3)^L_{0.5} = 2.1111$. From the state equation, we get

$(\dot{\tilde{r}}(t))^L_{0.5} = 2.1633$, then $(\tilde{r}(t))^L_{0.5} = 2.1633(t_F - t_o)$.

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Upper problem:

$$(I):\min_{(\tilde{\eta}_1)^U_{0.5}}\{[(\tilde{J}'_j)^U_{0.5}]=[1.5(\tilde{r})^U_{0.5}(t_F)-\int_{t_o}^{t_F}(0.5(\tilde{\eta}_1)^U_{0.5}-2.5)^2dt\}$$

$$(II):\min_{(\tilde{\eta}_2)^U_{0.5},(\tilde{\eta}_3)^U_{0.5}}\{(\tilde{J})^U_{0.5}=0.5(2.5(\tilde{r}(t_F))^U_{0.5}+3.5(\tilde{r}(t_F))^U_{0.5}-$$

$$\int_{t_o}^{t_F}((1.5(\tilde{\eta}_2)^U_{0.5}-3.5)^2+(0.5(\tilde{\eta}_3)^U_{0.5}-3.5)^2)dt)\}$$

subject to

$$(\dot{\tilde{r}}(t))^U_{0.5}=1.5(\tilde{\eta}_1)^U_{0.5}+1.5(\tilde{\eta}_2)^U_{0.5}+1.5(\tilde{\eta}_3)^U_{0.5}$$

Then, the Hamiltonian functions are:

$$(\tilde{H}')^U_{0.5}=-(0.5(\tilde{\eta}_1)^U_{0.5}-2.5)^2+$$
$$1.5(\tilde{p})^U_{0.5}((\tilde{\eta}_1)^U_{0.5}+(\tilde{\eta}_2)^U_{0.5}+(\tilde{\eta}_3)^U_{0.5}),$$

$$(\tilde{H})^U_{0.5}=-0.5[(1.5(\tilde{\eta}_2)^U_{0.5}-3.5)^2+(0.5(\tilde{\eta}_3)^U_{0.5}-3.5)^2]+$$
$$1.5(\tilde{q})^U_{0.5}((\tilde{\eta}_1)^U_{0.5}+(\tilde{\eta}_2)^U_{0.5}+(\tilde{\eta}_3)^U_{0.5})$$

For problem (I): Since $(\dot{\tilde{p}}(t))^U_{0.5}=-\dfrac{\partial(\tilde{H}')^U_{0.5}}{\partial(\tilde{r})^U_{0.5}}=0$, then $(\tilde{p}(t))^U_{0.5}=k$ (constant), and since $(\tilde{p}(t_F))^U_{0.5}=\dfrac{\partial(1.5(\tilde{r}(t_F))^U_{0.5})}{\partial(\tilde{r}(t_F))^U_{0.5}}=1.5$, then $(\tilde{p}(t))^U_{0.5}=1.5$.

Since $\dfrac{\partial(\tilde{H}')^U_{0.5}}{\partial(\tilde{\eta}_1)^U_{0.5}}=-2\times0.5(0.5(\tilde{\eta}_1)^U_{0.5}-2.5)+1.5(\tilde{p}(t))^U_{0.5}=0,$ then $(\tilde{\eta}_1)^U_{0.5}=9.5$.

For problem (II): Since $(\dot{\tilde{q}}(t))^U_{0.5}=-\dfrac{\partial(\tilde{H})^U_{0.5}}{\partial(\tilde{r})^U_{0.5}}=0$, then $(\tilde{q}(t))^U_{0.5}=k$ (constant), and since we have $(\tilde{q}(t_F))^U_{0.5}=\dfrac{\partial(3(\tilde{r}(t_F))^U_{0.5})}{\partial(\tilde{r}(t_F))^U_{0.5}}=3$, then $(\tilde{q}(t))^U_{0.5}=3$.

Since $\dfrac{\partial(\tilde{H})^U_{0.5}}{\partial(\tilde{\eta}_2)^U_{0.5}}=-1.5(1.5(\tilde{\eta}_2)^U_{0.5}-3.5)+1.5(\tilde{q}(t))^U_{0.5}=0$, then $(\tilde{\eta}_2)^U_{0.5}=4.333$.

Similarly $\dfrac{\partial(\tilde{H})^U_{0.5}}{\partial(\tilde{\eta}_3)^U_{0.5}}=-0.5(0.5(\tilde{\eta}_3)^U_{0.5}-3.5)+1.5(\tilde{q}(t))^U_{0.5}=0$, then $(\tilde{\eta}_3)^U_{0.5}=25$. From the state equation, $(\dot{\tilde{r}}(t))^U_{0.5}=58.249$, then $(\tilde{r}(t))^U_{0.5}=58.249(t_F-t_o)$.

Hence, the 0.5-NCO solution for this example is: $(\tilde{\eta}_1)_{0.5}=[1.0556,9.5]$, $(\tilde{\eta}_2)_{0.5}=[1.16,4.333]$ and $(\tilde{\eta}_3)_{0.5}=[2.111,25]$ with the correspondent state trajectory $(\tilde{r})_{0.5}=[2.1633,58.249](t_F-t_o)$.

VI. CONCLUSION

This paper discussed and presented a technique to get the optimal solution for the Nash-coalitive fuzzy continuous differential games (NCFDGs) problem in which the goals, state vectors and controls are fuzzy. The problem of NCFDGs was divided into two problems (lower and upper problems) by the aid of the α-cut set concept. In addition, the necessary and sufficient conditions for solving the NCFDGs were derived. An illustrative example was given.

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